

# $\mathbb{Q}$ -TRIVIAL GENERALIZED BOTT MANIFOLDS

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**ABSTRACT.** When the cohomology ring of a generalized Bott manifold with  $\mathbb{Q}$ -coefficient is isomorphic to that of a product of complex projective spaces  $\mathbb{C}P^{n_i}$ , the generalized Bott manifold is said to be  $\mathbb{Q}$ -trivial. We find a necessary and sufficient condition for a generalized Bott manifold to be  $\mathbb{Q}$ -trivial. In particular, every  $\mathbb{Q}$ -trivial generalized Bott manifold is diffeomorphic to a  $\prod_{n_i > 1} \mathbb{C}P^{n_i}$ -bundle over a  $\mathbb{Q}$ -trivial Bott manifold.

## 1. INTRODUCTION

A *generalized Bott tower of height  $h$*  is a sequence of complex projective space bundles

$$(1.1) \quad B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where  $B_i = P(\mathbb{C} \oplus \xi_i)$ ,  $\mathbb{C}$  is a trivial complex line bundle,  $\xi_i$  is a Whitney sum of  $n_i$  complex line bundles over  $B_{i-1}$ , and  $P(\cdot)$  stands a projectivization. Each  $B_i$  is called an  *$i$ -stage generalized Bott manifold*. When all  $n_i$ 's are 1 for  $i = 1, \dots, h$ , the sequence (1.1) is called a *Bott tower of height  $h$*  and  $B_i$  is called an  *$i$ -stage Bott manifold*.

A ( $h$ -stage) generalized Bott manifold is said to be  $\mathbb{Q}$ -trivial (respectively,  $\mathbb{Z}$ -trivial) if  $H^*(B_h; \mathbb{Q}) \cong H^*(\prod_{i=1}^h \mathbb{C}P^{n_i}; \mathbb{Q})$  (respectively,  $H^*(B_h; \mathbb{Z}) \cong H^*(\prod_{i=1}^h \mathbb{C}P^{n_i}; \mathbb{Z})$ ). It is shown in [CMS10b] that if  $B_h$  is  $\mathbb{Z}$ -trivial, then every fiber bundle in the tower (1.1) is trivial so that  $B_h$  is diffeomorphic to  $\prod_{i=1}^h \mathbb{C}P^{n_i}$ . Furthermore, Choi and Masuda show that every ring isomorphism between  $\mathbb{Z}$ -cohomology rings of two  $\mathbb{Q}$ -trivial Bott manifolds is induced by some diffeomorphism between them (see Theorem 3.1 and [CM09]).

We find a necessary and sufficient condition for a generalized Bott manifold to be  $\mathbb{Q}$ -trivial. Namely, we have the following proposition.

**Proposition 1.1.** *An  $h$ -stage generalized Bott manifold  $B_h$  is  $\mathbb{Q}$ -trivial if and only if each vector bundle  $\xi_i$ ,  $i = 1, \dots, h$ , satisfies*

$$(1.2) \quad (n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k$$

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for  $k = 1, \dots, n_i + 1$ , where  $B_i = P(\underline{\mathbb{C}} \oplus \xi_i)$ .

Moreover, the following theorem says that a  $\mathbb{Q}$ -trivial generalized Bott manifold without  $\mathbb{C}P^1$ -fibration is weakly equivariantly diffeomorphic to a trivial generalized Bott manifold.

**Theorem 1.2.** *Let  $B_h$  be a generalized Bott manifold such that all  $n_i$ 's are greater than 1. Then the following are equivalent*

- (1)  $B_h$  is  $\mathbb{Q}$ -trivial,
- (2) total Chern class  $c(\xi_i)$  is trivial for each  $i = 1, \dots, h$ ,
- (3)  $B_h$  is  $\mathbb{Z}$ -trivial, and
- (4)  $B_h$  is weakly equivariantly diffeomorphic to the product of projective spaces  $\prod_{i=1}^h \mathbb{C}P^{n_i}$ .

In the light of Theorem 1.2, we have a natural question.

**Question 1.3.** Let  $B_h$  and  $B'_h$  be generalized Bott manifolds with  $n_i > 1$ ,  $i = 1, \dots, h$ . Is  $H^*(B_h; \mathbb{Z})$  isomorphic to  $H^*(B'_h; \mathbb{Z})$  if  $H^*(B_h; \mathbb{Q}) \cong H^*(B'_h; \mathbb{Q})$ ?

Unfortunately, Example 3.7 shows that the answer to the question is negative.

From the proposition, we can deduce the following theorem.

**Theorem 1.4.** *Every  $\mathbb{Q}$ -trivial generalized Bott manifold is diffeomorphic to a  $\prod_{n_i > 1} \mathbb{C}P^{n_i}$ -bundle over a  $\mathbb{Q}$ -trivial Bott manifold.*

The remainder of this paper is organized as follows. In section 2, we recall general facts on a generalized Bott manifold and deal with its cohomology ring. In section 3, we prove Proposition 1.1, Theorem 1.2, and Theorem 1.4.

## 2. COHOMOLOGY RING OF A GENERALIZED BOTT MANIFOLD

Let  $B$  be a smooth manifold and let  $E$  be a complex vector bundle over  $B$ . Let  $P(E)$  denote the projectivization of  $E$ . Let  $y \in H^2(P(E))$  be the negative of the first Chern class of the tautological line bundle over  $P(E)$ . Then  $H^*(P(E))$  can be viewed as an algebra over  $H^*(B)$  via  $\pi^*: H^*(B) \rightarrow H^*(P(E))$ , where  $\pi: P(E) \rightarrow B$  denotes the projection. When  $H^*(B)$  is finitely generated and torsion free (this is the case when  $B$  is a toric manifold),  $\pi^*$  is injective and  $H^*(P(E))$  as an algebra over  $H^*(B)$  is known to be described as

$$(2.1) \quad H^*(P(E)) = H^*(B)[y] \left/ \left\langle \sum_{k=0}^n c_k(E) y^{n-k} \right\rangle \right.,$$

where  $n$  denotes the complex dimension of the fiber of  $E$  (see [BH58]).

For a generalized Bott manifold  $B_h$  in (1.1), since  $\pi_j^*: H^*(B_{j-1}) \rightarrow H^*(B_j)$  is injective, we regard  $H^*(B_{j-1})$  as a subring of  $H^*(B_j)$  for each  $j$  so that we have a filtration

$$H^*(B_h) \supset H^*(B_{h-1}) \supset \dots \supset H^*(B_1).$$

Let  $x_j \in H^2(B_j)$  denote minus the first Chern class of the tautological line bundle over  $B_j = P(\underline{\mathbb{C}} \oplus \xi_j)$ . We may think of  $x_j$  as an element of  $H^2(B_i)$

for  $i \geq j$ . Then the repeated use of (2.1) shows that the ring structure of  $H^*(B_h)$  can be described as

$$H^*(B_h) = \mathbb{Z}[x_1, \dots, x_h] \Big/ \left\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \dots + c_{n_i}(\xi_i)x_i \mid i = 1, \dots, h \right\rangle.$$

Let  $\xi_{2,1}$  be the tautological line bundle over  $B_1 = \mathbb{C}P^{n_1}$  and let  $\xi_{3,1} = \pi_2^*(\xi_{2,1})$  the pull-back bundle of the tautological line bundle over  $B_1$  to  $B_2$  via the projection  $\pi_2: B_2 \rightarrow B_1$ . In general, let  $\xi_{j,j-1}$  be the tautological line bundle over  $B_{j-1}$  and we define inductively

$$\xi_{j,j-k} = \pi_{j-1}^* \circ \dots \circ \pi_{j-k+1}^*(\xi_{j-k+1,j-k})$$

for  $k = 2, \dots, j-1$ . Then one can see that the Whitney sum of complex line bundles  $\xi_i$  over  $B_{i-1}$  in the sequence (1.1) can be written as

$$\xi_i := (\xi_{i,1}^{a_{i,1}^1} \otimes \dots \otimes \xi_{i,i-1}^{a_{i,i-1}^1}) \oplus \dots \oplus (\xi_{i,1}^{a_{i,1}^{n_i}} \otimes \dots \otimes \xi_{i,i-1}^{a_{i,i-1}^{n_i}})$$

for some integers  $a_{i,1}^1, \dots, a_{i,i-1}^1$ . Note that  $\xi_1 = (\mathbb{C})^{n_1}$ . Hence, the total Chern class of  $\xi_i$  is

$$(2.2) \quad c(\xi_i) = \prod_{j=1}^{n_i} \left( 1 + \sum_{k=1}^{i-1} a_{jk}^i x_k \right).$$

Therefore, the cohomology ring of  $B_h$  is

$$(2.3) \quad \begin{aligned} & H^*(B_h; \mathbb{Z}) \\ &= \mathbb{Z}[x_1, \dots, x_h] \Big/ \left\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \dots + c_{n_i}(\xi_i)x_i \mid i = 1, \dots, h \right\rangle \\ &= \mathbb{Z}[x_1, \dots, x_h] \Big/ \left\langle x_i \prod_{j=1}^{n_i} \left( \sum_{k=1}^{i-1} a_{jk}^i x_k + x_i \right) \mid i = 1, \dots, h \right\rangle. \end{aligned}$$

**Remark 2.1.** We can associate a generalized Bott manifold  $B_h$  with an  $h \times h$  vector matrix  $A$  as follows:

$$(2.4) \quad A^T = \begin{pmatrix} \mathbf{1} & & & \\ \mathbf{a}_1^2 & \mathbf{1} & & \\ \vdots & \vdots & \ddots & \\ \mathbf{a}_1^h & \mathbf{a}_2^h & \dots & \mathbf{1} \end{pmatrix},$$

where

$$\mathbf{a}_k^i = \begin{pmatrix} a_{1k}^i \\ \vdots \\ a_{n_i k}^i \end{pmatrix} \text{ and } \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Moreover we can consider  $B_h$  as a quasitoric manifold over the product of simplices  $\prod_{i=1}^h \Delta^{n_i}$  with the reduced characteristic matrix  $\Lambda_* = -A^T$ .

### 3. $\mathbb{Q}$ -TRIVIAL GENERALIZED BOTT MANIFOLDS

As we mentioned in the introduction, Choi and Masuda classify  $\mathbb{Q}$ -trivial Bott manifolds as follows.

**Theorem 3.1.** [CM09]

- (1) A Bott manifold  $B_h$  is  $\mathbb{Q}$ -trivial if and only if for each  $i = 1, \dots, h$ , each line bundle  $\xi_i$  satisfies  $c_1(\xi_i)^2 = 0$  in  $H^*(B_h; \mathbb{Z})$ .

- (2) Every ring isomorphism  $\varphi$  between two  $\mathbb{Q}$ -trivial Bott manifolds  $B_h$  and  $B'_h$  is induced by some diffeomorphism  $B_h \rightarrow B'_h$ .

In this section we shall prove Proposition 1.1 and Theorem 1.2. To prove them, we need the following lemmas.

**Lemma 3.2.** *If a generalized Bott manifold  $B_h$  is  $\mathbb{Q}$ -trivial, then there exist linearly independent primitive elements  $z_1, \dots, z_h$  in  $H^2(B_h; \mathbb{Z})$  such that  $z_i^{n_i}$  is not zero but  $z_i^{n_i+1}$  is zero in  $H^*(B_h; \mathbb{Z})$  for  $i = 1, \dots, h$ .*

*Proof.* Let  $H^*(B_h; \mathbb{Z})$  be generated by  $x_1, \dots, x_h$  as in (2.3) and let

$$H^*\left(\prod_{i=1}^h B_h; \mathbb{Q}\right) = \mathbb{Q}[y_1, \dots, y_h] / \left\langle y_i^{n_i+1} \mid i = 1, \dots, h \right\rangle.$$

Since both  $\{x_1, \dots, x_h\}$  and  $\{y_1, \dots, y_h\}$  are sets of generators of  $H^2(B_h; \mathbb{Q})$ , we can write

$$y_i = \sum_{j=1}^h c_{ij} x_j \quad \text{for } i = 1, \dots, h \text{ and } c_{ij} \in \mathbb{Q},$$

where the determinant of the matrix  $C = (c_{ij})_{h \times h}$  is non-zero. We may assume that  $c_{ij}$ 's are irreducible fractions. Multiplying  $(c_{i,1}, \dots, c_{i,h})$  by the least common denominator  $r_i$  of a set  $\{c_{i,1}, \dots, c_{i,h}\}$ , we can get a primitive element  $z_i = r_i y_i = r_i \sum_{j=1}^h c_{ij} x_j$  in  $H^2(B_h; \mathbb{Z})$  such that  $z_i^{n_i+1} = r_i^{n_i+1} y_i^{n_i+1}$  is zero in  $H^*(B_h; \mathbb{Z})$  for each  $i = 1, \dots, h$ . Since the elements  $y_1, \dots, y_h$  are linearly independent, the elements  $z_1, \dots, z_h$  are also linearly independent. Since  $y_i^{n_i}$  is not zero in  $H^*(B_h; \mathbb{Q})$ ,  $z_i^{n_i}$  cannot be zero in  $H^*(B_h; \mathbb{Z})$ . This proves the lemma.  $\square$

**Lemma 3.3.** [CMS10b] *Let  $B_m$  be an  $m$ -stage generalized Bott manifold. Then the set*

$$\{bx_m + w \in H^2(B_m) \mid 0 \neq b \in \mathbb{Z}, w \in H^2(B_{m-1}), (bx_m + w)^{n_m+1} = 0\}$$

*lies in a one-dimensional subspace of  $H^2(B_m)$  if it is non-empty.*

*Proof.* To satisfy  $(bx_m + w)^{n_m+1} = 0$ , we need  $bc_1(\xi_m) = (n_m + 1)w$ .  $\square$

**Lemma 3.4.** [CMS10b] *For an element  $z = \sum_{i=1}^h b_i x_i \in H^2(B_h)$ , if  $b_i$  is non-zero, then  $z^{n_i}$  cannot be zero in  $H^*(B_h)$ .*

*Proof.* If we expand  $(\sum_{i=1}^h b_i x_i)^{n_i}$ , there appears a non-zero scalar multiple of  $x_i^{n_i}$  because  $b_i \neq 0$ . Then,  $z^{n_i}$  cannot belong to the ideal generated by the polynomials  $x_i \prod_{j=1}^{n_i} (\sum_{k=1}^{i-1} a_{jk}^i x_k + x_i)$ , hence it is not zero in  $H^*(B_h)$ .  $\square$

Now we can prove Proposition 1.1.

*Proof of Proposition 1.1.* If each vector bundle  $\xi_i$  satisfies the conditions (1.2), then  $\left(x_i + \frac{1}{n_i+1} c_1(\xi_i)\right)^{n_i+1}$  is zero in  $H^*(B_h; \mathbb{Q})$ . Since the set

$$\left\{x_i + \frac{1}{n_i+1} c_1(\xi_i) \mid i = 1, \dots, h\right\}$$

generates  $H^*(B_h; \mathbb{Q})$  as a graded ring, this shows that  $B_h$  is  $\mathbb{Q}$ -trivial.

Conversely, if a generalized Bott manifold is  $\mathbb{Q}$ -trivial, then there are linearly independent and primitive elements  $z_1, \dots, z_h$  in  $H^2(B_h; \mathbb{Z})$  such that  $z_i^{n_i+1}$  is zero but  $z_i^{n_i}$  is not zero in  $H^*(B_h)$  by Lemma 3.2. We can put  $z_i = \sum_{j=1}^h b_{ij}x_j$  with  $b_{ij} \in \mathbb{Z}$  for each  $i = 1, \dots, h$ .

Now, consider a map  $\mu: \{1, \dots, h\} \rightarrow \mathbb{N}$  given by  $j \mapsto n_j$ . Further assume that the image of  $\mu$  is the set  $\{N_1, \dots, N_m\}$  with  $N_1 < \dots < N_m$ . We will show inductively that each  $z_i$  can be written as  $r_i \left( x_i + \frac{1}{\mu(i)+1} c_1(\xi_i) \right)$  for some  $r_i \in \mathbb{Z} \setminus \{0\}$ .

Case 1 : Assume  $i \in \mu^{-1}(N_1)$ . Let  $\mu^{-1}(N_1) := \{i_1, \dots, i_\alpha\}$  with  $i_1 < \dots < i_\alpha$ . We have  $z_i^{N_1+1} = 0$ . Then, by Lemma 3.4, we can see that

$$(3.1) \quad z_i = \sum_{j \in \mu^{-1}(N_1)} b_{ij}x_j,$$

that is,  $b_{ij'} = 0$  for  $j' \notin \mu^{-1}(N_1)$ . Note that for each  $i \in \mu^{-1}(N_1)$ , one of  $b_{ij}$ 's is nonzero for  $j \in \mu^{-1}(N_1)$  because the set  $\{z_i \mid i \in \mu^{-1}(N_1)\}$  is linearly independent. For some  $i_p \in \mu^{-1}(N_1)$ , if  $b_{i_p i_\alpha}$  is nonzero, then  $z_{i_p} \in H^2(B_{i_\alpha})$  and  $b_{ii_\alpha} = 0$  for all  $i \in \mu^{-1}(N_1) \setminus \{i_p\}$  by Lemma 3.3. Put  $w_{i_\alpha} := z_{i_p}$ . If  $b_{i_q i_{\alpha-1}}$  is nonzero for some  $i_q \in \mu^{-1}(N_1) \setminus \{i_p\}$ , then  $z_{i_q} \in H^2(B_{i_{\alpha-1}})$  and  $b_{ii_{\alpha-1}} = 0$  for all  $i \in \mu^{-1}(N_1) \setminus \{i_p, i_q\}$ . Now, put  $w_{i_{\alpha-1}} := z_{i_q}$ . In this way, for each  $i \in \mu^{-1}(N_1)$ , we can obtain  $w_i \in H^2(B_i)$  such that  $w_i \notin H^2(B_{i-1})$  and  $w_i^{N_1+1} = 0$  in  $H^*(B_h)$ . Moreover, from the proof of Lemma 3.3, we can write

$$(3.2) \quad w_i := r_i \left( x_i + \frac{1}{N_1 + 1} c_1(\xi_i) \right) \in H^2(B_i)$$

for each  $i \in \mu^{-1}(N_1)$ . In particular, if  $N_1 = 1$ , then  $w_i$  is of the form either  $\pm x_i$  or  $\pm(2x_i + c_1(\xi_i))$  for  $i \in \mu^{-1}(N_1)$ . Furthermore, without loss of generality, we may assume that  $z_i = w_i$  for  $i \in \mu^{-1}(N_1)$ .

Case 2 : Assume that  $z_k = r_k \left( x_k + \frac{1}{\mu(k)+1} c_1(\xi_k) \right)$  for  $N_1 \leq \mu(k) \leq N_{n-1}$  and let  $\ell \in \mu^{-1}(N_n)$ . Then we have  $z_\ell^{N_n+1} = 0$ . Then by Lemma 3.4, we can easily see that

$$z_\ell = \sum_{k \in \mu^{-1}(N_{<n})} b_{\ell k}x_k + \sum_{j \in \mu^{-1}(N_n)} b_{\ell j}x_j,$$

where  $N_{<n} = \{N_1, \dots, N_{n-1}\}$ . That is,  $b_{\ell j'} = 0$  for  $j' \notin \mu^{-1}(N_{\leq n})$ . Since  $z_\ell^{N_n+1}$  is zero in  $H^*(B_h)$ , we have

$$(3.3) \quad \begin{aligned} & \left( \sum_{k \in \mu^{-1}(N_{<n})} b_{\ell k}x_k + \sum_{j \in \mu^{-1}(N_n)} b_{\ell j}x_j \right)^{N_n+1} \\ &= \sum_{k \in \mu^{-1}(N_{<n})} f_k(x_1, \dots, x_h) (x_k^{\mu(k)+1} + c_1(\xi_k)x_k^{\mu(k)} + \dots + c_{\mu(k)}(\xi_k)x_k) \\ &+ \sum_{j \in \mu^{-1}(N_n)} b_{\ell j}^{N_n+1} (x_j^{N_n+1} + c_1(\xi_j)x_j^{N_n} + \dots + c_{N_n}(\xi_j)x_j) \end{aligned}$$

as polynomials, where  $f_k(x_1, \dots, x_h)$  is a homogeneous polynomial of degree  $N_n - \mu(k)$  for each  $k \in \mu^{-1}(N_{<n})$ . Note that for each  $\ell \in \mu^{-1}(N_n)$ , one of  $b_{\ell_j}$ 's is non-zero for  $j \in \mu^{-1}(N_n)$  from the linearly independency of the set  $\{z_i \mid i \in \mu^{-1}(N_{\leq n})\}$ . Let  $\mu^{-1}(N_n) := \{\ell_1, \dots, \ell_\beta\}$  with  $\ell_1 < \dots < \ell_\beta$ . Assume  $b_{\ell_p \ell_\beta}$  is nonzero for some  $\ell_p \in \mu^{-1}(N_n)$ . Substituting  $\ell = \ell_p$  into (3.3) and comparing the monomials containing  $x_{\ell_\beta}^{N_n}$  as a factor on both sides of (3.3), we have

$$(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_{<n})} b_{\ell_p k} x_k + \sum_{\substack{j \in \mu^{-1}(N_n) \\ j \neq \ell_\beta}} b_{\ell_p j} x_j \right) = b_{\ell_p \ell_\beta}^{N_n+1} c_1(\xi_{\ell_\beta}).$$

Since  $c_1(\xi_{\ell_\beta})$  belongs to  $H^2(B_{\ell_\beta-1})$ , we can see that  $b_{\ell_p k} = 0$  for  $k > \ell_\beta$ . That is,

$$z_{\ell_p} = \sum_{\substack{k \in \mu^{-1}(N_{<n}) \\ k < \ell_\beta}} b_{\ell_p k} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{\ell_p j} x_j.$$

Thus, we can see that  $z_{\ell_p} \in H^2(B_{\ell_\beta})$  and  $b_{\ell \ell_\beta} = 0$  for all  $\ell \in \mu^{-1}(N_n) \setminus \{\ell_p\}$  by Lemma 3.3. Put  $w_{\ell_\beta} := z_{\ell_p}$ . Now assume that  $b_{\ell_q \ell_{\beta-1}}$  is nonzero for some  $\ell_q \in \mu^{-1}(N_n) \setminus \{\ell_p\}$ . Substituting  $\ell = \ell_q$  into (3.3) and comparing the monomials containing  $x_{\ell_{\beta-1}}^{N_n}$  as a factor on both sides of (3.3), we have

$$(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_{<n})} b_{\ell_q k} x_k + \sum_{\substack{j \in \mu^{-1}(N_n) \\ j < \ell_{\beta-1}}} b_{\ell_q j} x_j \right) = b_{\ell_q \ell_{\beta-1}}^{N_n+1} c_1(\xi_{\ell_{\beta-1}}).$$

Since  $c_1(\xi_{\ell_{\beta-1}})$  belongs to  $H^2(B_{\ell_{\beta-1}-1})$ , we can see that  $b_{\ell_q k} = 0$  for  $k > \ell_{\beta-1}$ , and hence,

$$z_{\ell_q} = \sum_{\substack{k \in \mu^{-1}(N_{<n}) \\ k < \ell_{\beta-1}}} b_{\ell_q k} x_k + \sum_{\substack{j \in \mu^{-1}(N_n) \\ j < \ell_\beta}} b_{\ell_q j} x_j.$$

Thus, we can see that  $z_{\ell_q} \in H^2(B_{\ell_{\beta-1}})$  and  $b_{\ell \ell_{\beta-1}} = 0$  for all  $\ell \in \mu^{-1}(N_n) \setminus \{\ell_p, \ell_q\}$  by Lemma 3.3. Now, put  $w_{\ell_{\beta-1}} := z_{\ell_q}$ . In this way, for each  $\ell \in \mu^{-1}(N_n)$ , we can obtain  $w_\ell \in H^2(B_\ell)$  such that  $w_\ell \notin H^2(B_{\ell-1})$  and  $w_\ell^{N_n+1} = 0$  in  $H^2(B_h)$ . Moreover, from the proof of Lemma 3.3,  $w_\ell$  can be written as  $r_\ell \left( x_\ell + \frac{1}{N_n+1} c_1(\xi_\ell) \right)$ . Furthermore, without loss of generality, we may assume that  $z_\ell = w_\ell$  for  $\ell \in \mu^{-1}(N_n)$ .

By Cases 1 and 2, we can see that, for each  $i = 1, \dots, h$ , we can write

$$z_i = r_i \left( x_i + \frac{1}{n_i + 1} c_1(\xi_i) \right)$$

for some  $r_i \in \mathbb{Z} \setminus \{0\}$ . Therefore,  $\{(n_i + 1)x_i + c_1(\xi_i)\}^{n_i+1}$  is zero in  $H^*(B_h)$ . From this, we can see

$$(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k \text{ and } c_1(\xi_i)^{n_i+1} = 0$$

$k = 1, \dots, n_i$ . □

Hence, the above theorem implies the statement (1) of Theorem 3.1.

By using Proposition 1.1, we can prove Theorem 1.2.

*Proof of Theorem 1.2.* We first prove the implication (1) $\Rightarrow$ (2). By Proposition 1.1, we have the relation

$$(3.4) \quad (n_i + 1)^2 c_2(\xi_i) = \frac{n_i(n_i + 1)}{2} c_1(\xi_i)^2.$$

If  $n_i = 2$ , from (2.2) and (3.4), we have

$$(3.5) \quad \begin{aligned} & \{(a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1}) + (a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1})\}^2 \\ &= 3(a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1})(a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1}). \end{aligned}$$

For  $j = 1, \dots, i-1$ , since  $x_j^2 \neq 0$  in  $H^*(B_i)$ , by comparing the coefficients of  $x_j^2$  on both sides of (3.5), we have  $(a_{1j}^i + a_{2j}^i)^2 = 3a_{1j}^i a_{2j}^i$  whose integer solution is only  $a_{1j}^i = a_{2j}^i = 0$ . If  $n_i = n > 2$ , then we have

$$(3.6) \quad \begin{aligned} & n\{(a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1}) + \cdots + (a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1})\}^2 \\ &= 2(n+1)\{(a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1})(a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1}) \\ &\quad + \cdots + (a_{n-1,1}^i x_1 + \cdots + a_{n-1,i-1}^i x_{i-1})(a_{n,1}^i x_1 + \cdots + a_{n,i-1}^i x_{i-1})\}. \end{aligned}$$

Since  $x_j^2 \neq 0$  in  $H^*(B_i)$  for  $j = 1, \dots, i-1$ , by comparing the coefficients of  $x_j^2$  on both sides of (3.6) we have

$$(3.7) \quad n(a_{i,j}^i + \cdots + a_{n,j}^i)^2 = 2(n+1) \sum_{1 \leq k < \ell \leq n} a_{kj}^i a_{\ell j}^i.$$

The equation (3.7) is equivalent to

$$(n-2)\{(a_{1j}^i)^2 + \cdots + (a_{nj}^i)^2\} + (a_{1j}^i - a_{2j}^i)^2 + \cdots + (a_{n-1,j}^i - a_{nj}^i)^2 = 0.$$

Since  $n > 2$ , we can see that  $a_{1j}^i = \cdots = a_{nj}^i = 0$  for each  $j = 1, \dots, i-1$ . Therefore, in any case,  $c(\xi_i)$  is trivial for all  $i = 1, \dots, h$ .

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear.

The implication (3) $\Leftrightarrow$ (4) is proved by Choi-Masuda-Suh [CMS10b].

Therefore, all four conditions are equivalent.  $\square$

From the proof of (1) $\Rightarrow$ (2), we have the following corollary.

**Corollary 3.5.** *A  $\mathbb{Q}$ -trivial generalized Bott manifold  $B_h$  is weakly equivariantly diffeomorphic to  $\prod_{i=1}^h \mathbb{C}P^{n_i}$  provided  $n_i > 1$  for all  $i$ .*

*Proof.* Since for each  $i = 1, \dots, h$ ,  $a_{1j}^i = \cdots = a_{nj}^i = 0$  for all  $j = 1, \dots, i-1$ . Hence, the associated vector matrix of  $B_h$  is block diagonal. Hence, the assertion is true.  $\square$

From Theorem 1.2, we have the following corollary.

**Corollary 3.6.** *Let  $M$  be a quasitoric manifold. If  $H^*(M; \mathbb{Q})$  is isomorphic to  $H^*(\prod_{i=1}^h \mathbb{C}P^{n_i}; \mathbb{Q})$ , then  $M$  is homeomorphic to  $\prod_{i=1}^h \mathbb{C}P^{n_i}$  provided  $n_i > 1$  for all  $i$ .*

*Proof.* By [CMS10a], if  $H^*(M; \mathbb{Q})$  is isomorphic to  $H^*(\prod_{i=1}^h \mathbb{C}P^{n_i}; \mathbb{Q})$ , then  $M$  is homeomorphic to a generalized Bott manifold. But a  $\mathbb{Q}$ -trivial generalized Bott manifolds with  $n_i > 1$  is diffeomorphic to  $\prod_{i=1}^h \mathbb{C}P^{n_i}$ . Hence,  $M$  is homeomorphic to  $\prod_{i=1}^h \mathbb{C}P^{n_i}$ .  $\square$

The following is the counter-example of Question 1.3.

**Example 3.7.** Let  $B$  be a fiber bundle  $P(\underline{\mathbb{C}}^3 \oplus \xi)$  over  $\mathbb{C}P^2$  and let  $B'$  be a fiber bundle  $P(\underline{\mathbb{C}}^3 \oplus \xi^{\otimes 2})$  over  $\mathbb{C}P^2$ , where  $\xi$  is the tautological line bundle over  $\mathbb{C}P^2$ . Let  $y$  (respectively,  $Y$ ) denote the negative of the first Chern class of the tautological line bundle over  $B_2$  (respectively,  $B'_2$ ). Then their cohomology rings are

$$H^*(B) = \mathbb{Z}[x, y] / \langle x^3, y(y^3 + xy^2) \rangle$$

and

$$H^*(B') = \mathbb{Z}[X, Y] / \langle X^3, Y(Y^3 + 2XY^2) \rangle.$$

Then the map  $\phi$  defined by  $\phi(x) = 2X$  and  $\phi(y) = Y$  is an isomorphism from  $H^*(B; \mathbb{Q}) \rightarrow H^*(B'; \mathbb{Q})$ . But this  $\phi$  is not a  $\mathbb{Z}$ -isomorphism. Suppose that  $\psi$  is an isomorphism  $H^*(B; \mathbb{Z}) \rightarrow H^*(B'; \mathbb{Z})$ . Then there exist  $\alpha, \beta, \gamma, \delta$  in  $\mathbb{Z}$  such that

$$\begin{pmatrix} \psi(x) \\ \psi(y) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and  $\alpha\delta - \beta\gamma = \pm 1$ . Since  $\psi(x^3) = 0$  in  $H^*(B'; \mathbb{Z})$ , we have

$$(\alpha X + \beta Y)^3 = \alpha^3 X^3$$

as polynomials. So, we can see that  $\beta$  is zero and  $\alpha = \pm 1$ , and hence  $\delta = \pm 1$ . Since  $\psi(y(y^3 + xy^2))$  is zero in  $H^*(B'; \mathbb{Z})$ , we have

$$(3.8) \quad (\gamma X + \delta Y)^3((\alpha + \gamma)X + \delta Y) = (aX + bY)X^3 + cY(Y^3 + 2XY^2)$$

as polynomials in  $\mathbb{Z}[X, Y]$ . By comparing the coefficients of  $XY^3$  on both sides of (3.8), we can see that

$$(3.9) \quad 2c = 3\gamma\delta^3 + (\alpha + \gamma)\delta^3 = \delta(\alpha + 4\gamma).$$

Since the right hand side of (3.9) is odd, there is no such an integer  $C$ . Hence, there is no such  $\mathbb{Z}$ -isomorphism  $\psi$ .

Now consider  $\mathbb{Q}$ -trivial generalized Bott manifolds  $B_h$  which have  $\mathbb{C}P^1$ -fibers, that is,  $n_k = 1$  for some  $k \in [h]$ .

**Lemma 3.8.** *Let  $B_h$  and  $B'_h$  be two  $h$ -stage generalized Bott towers. If the associated vector matrices to them are*

$$A = \begin{pmatrix} \mathbf{1} & & & & \\ * & \ddots & & & \\ * & * & \mathbf{1} & & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{h-2} & \mathbf{1} & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_{h-2} & \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} \mathbf{1} & & & & \\ * & \ddots & & & \\ * & * & \mathbf{1} & & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_{h-2} & \mathbf{1} & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{h-2} & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

respectively, then  $B_h$  and  $B'_h$  are equivariantly diffeomorphic.



*Proof.* Note that this lemma can be seen by the fact that  $B_h$  and  $B'_h$  are equivariantly diffeomorphic if two associated vector matrices are conjugated by a permutation matrix, see the paper [CMS10a]. It is obvious that

$$A' = E_\sigma A E_\sigma^{-1},$$

where  $\sigma := (1, \dots, h-2, h, h-1)$  is the permutation on  $[h]$  which permutes only  $h-1$  and  $h$ .  $\square$

Now, we can prove Theorem 1.4.

*Proof of Theorem 1.4.* Let  $B_h$  be a  $\mathbb{Q}$ -trivial generalized Bott manifold whose associated matrix is of the form (2.4).

Consider a map  $\mu: \{1, \dots, h\} \rightarrow \mathbb{N}$  given by  $j \mapsto n_j$  and assume that the image of  $\mu$  is the set  $\{N_1, \dots, N_m\}$  with  $1 = N_1 < N_2 < \dots < N_m$ .

For each  $i \in \mu^{-1}(1)$ , by Proposition 1.1, we have  $c_1(\xi_i)^2 = 0$  in  $H^*(B_h)$ . Since  $x_k^2 \neq 0$  in  $H^*(B_h)$  for  $k \notin \mu^{-1}(1)$ , we can see that  $a_{1k}^i = 0$  for  $k \in [i-1]$  with  $n_k > 1$ .

Now suppose that  $n_j > 1$ . Then by Proposition 1.1, we have the relation

$$(n_j + 1)^2 c_2(\xi_j) = \frac{n_j(n_j + 1)}{2} c_1(\xi_j)^2.$$

Since  $x_k^2 \neq 0$  in  $H^*(B_h)$  for  $n_k > 1$ , we can show that  $\mathbf{a}_k^j = \mathbf{0}$  by using the same argument to the proof of Theorem 1.2.

Since  $\mathbf{a}_k^j = \mathbf{0}$  for all  $n_k > 1$ , by Lemma 3.8,  $B_h$  is diffeomorphic to the  $\mathbb{Q}$ -trivial generalized Bott manifold  $B'$  whose associated matrix is of the form

$$(3.10) \quad (A')^T = \left( \begin{array}{cccc|cc} 1 & & & & & \\ a_{11}^2 & 1 & & & & \\ \vdots & \vdots & \ddots & & & \\ a_{11}^r & a_{1,2}^r & \cdots & 1 & & \\ \hline \mathbf{a}_1^{r+1} & \mathbf{a}_2^{r+1} & \cdots & \mathbf{a}_r^{r+1} & \mathbf{1} & \\ \mathbf{a}_1^{r+2} & \mathbf{a}_2^{r+2} & \cdots & \mathbf{a}_r^{r+2} & \mathbf{0} & \mathbf{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \\ \mathbf{a}_1^h & \mathbf{a}_2^h & \cdots & \mathbf{a}_r^h & \mathbf{0} & \cdots \mathbf{0} \mathbf{1} \end{array} \right),$$

where  $r$  is the cardinality of the set  $\mu^{-1}(1)$ , that is,  $r = |\mu^{-1}(1)|$ . This proves the theorem.  $\square$

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